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Hybrid discretization method for time-delay nonlinear systems †

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Abstract

A hybrid discretization scheme that combines the virtues of the Taylor series and Matrix exponential integration methods is proposed. In the algorithm, each sampling time interval is divided into two subintervals to be considered according to the time delay and sampling period. The algorithm is not too expensive computationally and lends itself to be easily inserted into large simulation packages. The mathematical structure of the new discretization scheme is explored and described in detail. The performance of the proposed discretization procedure is evaluated by employing case studies. Various input signals, sampling rates, and time-delay values are considered to test the proposed method. The results demonstrate that the proposed discretization scheme is better than previous Taylor series method for nonlinear time-delay systems, especially when a large sampling period is inevitable.

Keywords: Nonlinear system; Hybrid discretization algorithm; Matrix exponential; Time delay; Taylor series

1. Introduction

Whenever material, information, or energy is physically transmitted from one place to another, delay is associated with the transmission. The value of the delay is determined by the distance and transmission speed. Some delays are short, while others are very long. The presence of delays (especially, long delays) renders system analysis and control design much more complex [1].

Systems with delays abound in the world. They appear in various systems such as biological, ecological, economic, social, and engineering systems. Among the typical examples of time-delay systems are communication networks, chemical processes, teleoperation systems, biosystems, and underwater vehicles [2].

Engineering studies dealing with time-delay systems are extensive. Thus far, most proposed approaches have dealt with linear time-delay control systems and, in particular, with stability analysis and behavior of such systems with constant and/or uncertain time delays [3-7].

However, it should be mentioned that conventional numerical techniques such as the Euler and Runge-Kutta methods have been employed in order to obtain a sampled-data representation of the original continuous-time delay-free system [8]. All these approaches require a very small time-step in order to be deemed accurate. This may not be the case in control applications where large sampling periods are inevitably introduced due to physical and technical limitations. Furthermore, too many integration steps may cause unacceptable integration times and even excessive error accumulation.

F. Allffi-Pentini et al. proposed a novel algorithm for the numerical integration of systems of ordinary differential equations [9]. Their method has several features. First, it preserves the virtues of a matrix-exponential solver with step length control (i.e., it is largely robust to ill-conditioning). It is suitable for any nonlinear problem and provides the exact analytic solution to linear problems. Second, the method for the computation of the matrix exponential does not deliberately neglect the effects of any eigenvalue. Third, it includes a novel algorithm for the automatic correction of the accumulation effect of some kinds of rounding errors.

In large sampling period systems, the Taylor series method has also been used to improve the performance of the controller [10]. The proposed discretization method is based on the Taylor series and uses a mathematical framework similar to one previously developed for delay-free nonlinear systems [11-15].

However, the matrix exponential method is time-consuming and the precision of Taylor series is not sufficient for certain applications. In order to combine the virtues of these two methods and design a method for time-delay system, a hybrid scheme is proposed.

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In particular, this paper makes the following contributions:

A hybrid discretization method based on the famous Taylor series and Matrix exponential is proposed. It retains all the merits of the Taylor series and Matrix exponential. This new method is suitable for any nonlinear input time-delay system, especially for large sampling period systems with large time delay.

The algorithm of the new scheme is described in detail and its usage is explained through various simulations.

The results of the simulations illustrate that the new scheme is applicable.

This paper is organized as follows: Section 2 contains some mathematical preliminaries. Section 3 briefly introduces the matrix exponential and Taylor series discretization methods. The hybrid discretization method is then proposed and described in detail. Case studies are presented in Section 4, where two different kinds of system are inspected using various sampling rates, time delays, and input signals. Finally, Section 5 provides a few concluding remarks drawn from this study.

2. Preliminaries

In the present study, single-input nonlinear continuous time control systems are considered with a state-space representation of the form

$$\frac{dx(t)}{dt} = f(x(t)) + g(x(t))u(t-D) \tag{1}$$

where $x \in X \subset R^n$ is the vector of the states representing an open and connected set; $u \in R$ is the input variable; and D is the system's constant time delay (dead-time) that directly affects the input. It is assumed that $f(x)$ and $g(x)$ are real analytic vector fields on X.

An equidistant grid on the time axis with mesh $T = t_{k+1} - t_k > 0$ is considered, where $[t_k, t_{k+1}) = [kT, (k+1)T)$ is the sampling interval and T is the sampling period. It is also assumed that System (1) is driven by an input that is piecewise constant over the sampling interval (i.e., the zero-order hold (ZOH) assumption holds true).

For the ZOH assumption, while $D = 0$,

$$u(t) = u(kT) \equiv u(k) = constant \tag{2}$$

for $kT \le t < kT + T$. Furthermore, let

$$D = qT + \gamma \tag{3}$$

where $q \in \{0,1,2,...\}$ and $0 < \gamma < T$. Equivalently, the time delay D is customarily represented as an integer multiple of the sampling period plus a fractional part of T. Under the ZOH assumption and the above notation, it is rather straightforward to verify that if "delayed" input variable attains the following values with expressions within the sampling interval, while $D \neq 0$,

$$u(t-D) = \begin{cases} u(kT - qT - T) \equiv u(k-q-1); & t \in [\mathrm{kT}, \mathrm{kT}+\gamma) \\ u(kT - qT) \equiv u(k-q); & t \in [\mathrm{kT}+\gamma, \mathrm{kT}+T) \end{cases} \tag{4}$$

3. Hybrid discretization scheme

In this section, a hybrid discretization method based on a brief introduction to the matrix exponential and Taylor series discretization methods is proposed, and the corresponding programming algorithm is described in detail.

3.1 Matrix exponential discretization method

First, delay-free ($D = 0$) nonlinear control systems are considered with a state-space representation of the form,

$$\frac{dx(t)}{dt} = f(x(t)) + g(x(t))u(t) \tag{5}$$

Consider the time interval $[t_k, t_{k+1})$, and suppose

$$u(t) = u_k, \quad t \in [t_k, t_{k+1}).$$

Let us denote,

$$\xi(t) = x(t) - x_k \tag{6}$$

where $t \in [t_k, t_{k+1})$, $x_k = x_{t=kT}$.

We then get the following approximation

$$f(x(t)) \approx f(x_k) + \frac{\partial f(x_k)}{\partial x}\xi(t) \tag{7}$$

$$g(x(t)) \approx g(x_k) + \frac{\partial g(x_k)}{\partial x}\xi(t). \tag{8}$$

From (6), we obtain

$$\dot{\xi}(t) = \dot{x}(t), \quad \xi(t_k) = 0 \tag{9}$$

Based on the above, within the time interval $[t_k, t_{k+1})$, System (5) can be approximated through the following expression:

$$\begin{aligned} \dot{\xi}(t) &= f(x_k) + \frac{\partial f(x_k)}{\partial x}\xi(t) + (g(x_k) + \frac{\partial g(x_k)}{\partial x}\xi(t))u_k \\ &= (f(x_k) + g(x_k)u_k) + (\frac{\partial f(x_k)}{\partial x} + \frac{\partial g(x_k)}{\partial x}u_k)\xi(t) \\ &= \tilde{f}_k + J_k\xi(t) \end{aligned} \tag{10}$$

where

$$\tilde{f}_k = \tilde{f}_\xi(x_k, u_k) = f(x_k) + g(x_k)u_k \tag{11}$$

$$J_k = J_\xi(x_k, u_k) = \frac{\partial f(x_k)}{\partial x} + \frac{\partial g(x_k)}{\partial x}u_k \tag{12}$$

Rewriting Eq. (10), we obtain

$$\dot{\xi}(t) = \tilde{f}_k + J_k \xi(t), \quad \xi(\mathrm{t_k})=0 \tag{13}$$

with $N > 0$ being an integer number, we denote

$$h_k = \frac{t_{k+1} - t_k}{N} \tag{14}$$

Let us introduce an extended vector,

$$\eta(t) = \begin{pmatrix} \xi(t) \\ 1 \end{pmatrix} \tag{15}$$

System (13) can then be rewritten in the form

$$\dot{\eta}(t) = C_k \eta(t), \ \ \eta(\mathrm{t_k}) = \begin{pmatrix} \overline{0} \\ 1 \end{pmatrix} = \eta_0 \tag{16}$$

where $C_k = \begin{pmatrix} J_k & \tilde{f}_k \\ \overline{0}^T & 0 \end{pmatrix} \in R^{(n+1)\times(n+1)}$, $\overline{0}$ is an n dimensional zero column vector and $\overline{0}^T$ is an n dimensional zero row vector.

By solving (16), we obtain

$$\eta(t_{k+1}) = e^{c_k \cdot (t_{k+1} - t_k)} \eta_0 \tag{17}$$

To calculate the matrix exponential, let Z be a square matrix and I the corresponding identity matrix. The exact formula is

$$e^Z = \lim_{N\to\infty} \left(I + \frac{Z}{N} \right)^N \tag{18}$$

Eq. (18) provides the truncated approximation

$$e^Z \cong \left(I + \frac{Z}{N} \right)^N \tag{19}$$

for a suitable value of *N*.

An improved form is

$$e^Z \cong \left(I + \frac{Z}{2^b} \right)^{2^b} \tag{20}$$

for a suitable value of *b*, since it can be computed by *b* iterated squaring operations starting from $I + \frac{Z}{2^b}$. The method of choosing a suitable value for the exponent *b* was introduced by F. Allffi-Pentini, et al [9].

By combining (19) and (17), we obtain

$$e^{c_k \cdot (t_{k+1} - t_k)} \approx \left(I + \frac{C_k (t_{k+1} - t_k)}{N} \right)^N = \left(I + C_k h_k \right)^N \tag{21}$$

From (21) and (17), we obtain

$$\eta(t_{k+1}) = \left(I + C_k h_k \right)^N \eta_0 \Rightarrow$$

$$\eta(t_{k+1}) = \begin{pmatrix} I + h_k J_k & h_k \tilde{f}_k \\ \overline{0}^T & 1 \end{pmatrix}^N \begin{pmatrix} \overline{0} \\ 1 \end{pmatrix} \Rightarrow$$

$$\xi(t_{k+1}) = \begin{pmatrix} I & \overline{0} \end{pmatrix} \begin{pmatrix} I + h_k J_k & h_k \tilde{f}_k \\ \overline{0}^T & 1 \end{pmatrix}^N \begin{pmatrix} \overline{0} \\ 1 \end{pmatrix} \tag{22}$$

where $\begin{pmatrix} I & \overline{0} \end{pmatrix} \in R^{n\times(n+1)}$.

Hence,

$$x_{k+1} = x_k + \begin{pmatrix} I & \overline{0} \end{pmatrix} \begin{pmatrix} I + h_k J_k & h_k \tilde{f}_k \\ \overline{0}^T & 1 \end{pmatrix}^N \begin{pmatrix} \overline{0} \\ 1 \end{pmatrix} \tag{23}$$

Second, let us consider the time-delay system (1). Since the time delay is introduced, we should consider each sampling time interval $[kT, kT+T)$ as two subintervals, $[kT, kT+\gamma)$ and $(kT+\gamma, kT+T]$.

To apply the discretization method of (23), we should first choose N_1 and N_2, and make them meet the requirement

$$\frac{N_1}{N_2} \approx \frac{\gamma}{T-\gamma} \tag{24}$$

where N_1 and N_2 are both positive natural numbers. Eq. (24) indicates that the calculation step lengths of $[kT, kT+\gamma)$ and $(kT+\gamma, kT+T]$ are nearly identical (see also (26) and (30)).
Applying (23) for the subinterval $[kT, kT+\gamma)$, we get

$$x_{kT+\gamma} = x_k + \begin{pmatrix} I & \overline{0} \end{pmatrix} \begin{pmatrix} I + h_{k_1} J_{k_1} & h_{k_1} \tilde{f}_{k_1} \\ \overline{0}^T & 1 \end{pmatrix}^{N_1} \begin{pmatrix} \overline{0} \\ 1 \end{pmatrix} \tag{25}$$

where

$$h_{k_1} = \frac{\gamma}{N_1}, \tag{26}$$

$$\tilde{f}_{k_1} = \tilde{f}_\xi (x(kT), u(k-q-1)), \tag{27}$$

$$J_{k_1} = J_\xi (x(kT), u(k-q-1)). \tag{28}$$

Applying (23) for the subinterval $(kT+\gamma, kT+T]$, we get

$$x_{k+1} = x_{kT+T} = x_{kT+\gamma} + \begin{pmatrix} I & \overline{0} \end{pmatrix} \begin{pmatrix} I + h_{k_2} J_{k_2} & h_{k_2} \tilde{f}_{k_2} \\ \overline{0}^T & 1 \end{pmatrix}^{N_2} \begin{pmatrix} \overline{0} \\ 1 \end{pmatrix} \tag{29}$$

where

$$h_{k_2} = \frac{T-\gamma}{N_2}, \tag{30}$$

$$\tilde{f}_{k_2} = \tilde{f}_\xi (x(kT+\gamma), u(k-q)), \tag{31}$$

$$J_{k_2} = J_\xi (x(kT+\gamma), u(k-q)). \tag{32}$$

Combining (25) and (29), we obtain

$$x_{k+1} = x_k + \begin{pmatrix} I & \bar{0} \end{pmatrix} \begin{pmatrix} I + h_{k_1} J_{k_1} & h_{k_1} \tilde{f}_{k_1} \\ \bar{0}^T & 1 \end{pmatrix}^{N_1} \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix}$$
$$+ \begin{pmatrix} I & \bar{0} \end{pmatrix} \begin{pmatrix} I + h_{k_2} J_{k_2} & h_{k_2} \tilde{f}_{k_2} \\ \bar{0}^T & 1 \end{pmatrix}^{N_2} \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix} \tag{33}$$

The state of System (1) at moment $t = kT + \gamma$ can be expressed as follows:

$$x(kT + \gamma) = \Phi_{\gamma, N_1}(x(kT), u(k - q - 1)) \tag{34}$$

The state at moment $t = kT + T$ can be expressed as

$$x(kT + T) = \Phi_{T-\gamma, N_2}(x(kT + \gamma), u(k - q)) \tag{35}$$

Consequently, we have

$$\begin{aligned} x(k+1) &= \Phi_{T-\gamma, N_2}(\Phi_{\gamma, N_1}(x(k), u(k-q-1)), u(k-q)) \\ &= \Psi_{T,\gamma,N_1,N_2}(x(k), u(k-q-1), u(k-q)) \\ &= \Psi_*(x(k), u(k-q-1), u(k-q)) \end{aligned} \tag{36}$$

Remark 1: The special case where $\gamma = 0$ and $D = qT$ frequently occurs in practice when modeling and designing digital control systems. In this case, we easily obtain

$$\begin{aligned} x_{k+1} &= x_k + \begin{pmatrix} I & \bar{0} \end{pmatrix} \begin{pmatrix} I + h_k J_k & h_k \tilde{f}_k \\ \bar{0}^T & 1 \end{pmatrix}^{N} \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix} \\ &= \Phi_{T,N}(x(k), u(k-q)) \end{aligned} \tag{37}$$

3.2 Taylor series-based discretization method

Initially, within a sampling interval, the solution of (5) is expanded in a uniformly convergent Taylor series. The resulting coefficients can be easily computed by taking successive partial derivatives of the right hand-side of (5).

$$\begin{aligned} &x(k+1) \\ &= x(k) + \sum_{\ell=1}^{\infty} \frac{T^\ell}{\ell!} \frac{d^\ell x}{dt^\ell}\bigg|_{t_k} = x(k) + \sum_{\ell=1}^{\infty} A^{[\ell]}(x(k), u(k)) \frac{T^\ell}{\ell!} \end{aligned} \tag{38}$$

where $x(k)$ is the value of the state vector x at time $t = t_k = kT$ and $A^{[\ell]}(x,u)$ are determined recursively by

$$\begin{cases} A^{[1]}(x,u) = f(x) + ug(x) \\ A^{[\ell+1]}(x,u) = \dfrac{\partial A^{[\ell]}(x,u)}{\partial x}(f(x) + ug(x)) \end{cases} \tag{39}$$

where $\ell = 1, 2, 3 \ldots$.

Therefore, an approximate sampled-data representation (ASDR) of Eq. (5) is obtained from the truncation of the Taylor series of order N,

$$\begin{aligned} x(k+1) &= \Phi_T^N(x(k), u(k)) \\ &= x(k) + \sum_{\ell=1}^{N} A^{[\ell]}(x(k), u(k)) \frac{T^\ell}{\ell!} \end{aligned} \tag{40}$$

where the subscript T of the mapping Φ_T^N denotes the dependence on the sampling period, and the superscript N denotes the finite series truncation order associated with the ASDR of Eq. (40).

By applying the Taylor series discretization method for nonlinear systems (1) to the $[kT, kT+\gamma)$ subinterval, one immediately obtains the state vector evaluated at $kT + \gamma$,

$$x(kT + \gamma) = \Phi_\gamma(x(kT), u(k - q - 1)) \tag{41}$$

where the map Φ_γ can be derived through the direct application of Formula (38), and the subsequent calculation of the corresponding Taylor coefficients can be realized by employing the recursive formulas in (39). $x(kT)$ and $u(k-q-1)$ are the instantaneous state vector and input value, respectively, at time kT.

Similarly, the application of the Taylor discretization method to the $[kT+\gamma, kT+T)$ subinterval yields the state vector evaluated at $(k+1)T$ as a function of $x(kT+\gamma)$ and the input value at time $kT + \gamma$,

$$x(kT + T) = \Phi_{T-\gamma}(x(kT + \gamma), u(k - q)) \tag{42}$$

Based on (40), Eqs. (41) and (42) can be rewritten as follows:

$$x(kT + \gamma) = x(kT) + \sum_{\ell=1}^{\infty} A^{[\ell]}(x(kT), u(k-q-1)) \frac{\gamma^\ell}{\ell!} \tag{43}$$

$$\begin{aligned} x(kT + T) &= x(kT + \gamma) \\ &+ \sum_{\ell=1}^{\infty} A^{[\ell]}(x(kT+\gamma), u(k-q)) \frac{(T-\gamma)^\ell}{\ell!} \end{aligned} \tag{44}$$

By combining (43) and (44), the desired sampled data representation of a time-delay nonlinear system is obtained,

$$\begin{aligned} x(k+1) &= \Phi_T^D(x(k), u(k-q-1), u(k-q)) \\ &= \Phi_{T-\gamma}(\Phi_\gamma(x(k), u(k-q-1)), u(k-q)) \end{aligned} \tag{45}$$

Therefore, the finite series truncation order N for the associated Taylor series (45) is shown below as an ASDR,

$$x(k+1) = \Phi_T^{N,D}(x(k), u(k-q-1), u(k-q)) \tag{46}$$

3.3 Hybrid discretization scheme

Based on the above two discretization methods, we propose a novel hybrid discretization scheme, which is described as follows.

First, for delay-free ($D = 0$) nonlinear control systems described by (5), based on the zero-order hold assumption, we considered the time interval $[t_k, t_{k+1})$.

Then, according to the theory of the Taylor series and (6), we obtained the following approximation:

$$f(x(t)) \approx f(x_k) + \frac{\partial f(x_k)}{\partial x}\xi(t) + \frac{1}{2}\frac{\partial^2 f(x_k)}{\partial x^2}\xi^2(t) + \ldots + \frac{1}{n!}\frac{\partial^n f(x_k)}{\partial x^n}\xi^n(t) \quad (47)$$

$$g(x(t)) \approx g(x_k) + \frac{\partial g(x_k)}{\partial x}\xi(t) + \frac{1}{2}\frac{\partial^2 g(x_k)}{\partial x^2}\xi^2(t) + \ldots + \frac{1}{n!}\frac{\partial^n g(x_k)}{\partial x^n}\xi^n(t) \quad (48)$$

To simplify, we set the truncation order of (47) and (48) as *2*, as follows:

$$f(x(t)) \approx f(x_k) + \frac{\partial f(x_k)}{\partial x}\xi(t) + \frac{1}{2!}\frac{\partial^2 f(x_k)}{\partial x^2}\xi^2(t) \quad (49)$$

$$g(x(t)) \approx g(x_k) + \frac{\partial g(x_k)}{\partial x}\xi(t) + \frac{1}{2!}\frac{\partial^2 g(x_k)}{\partial x^2}\xi^2(t) \quad (50)$$

Based on (9), (49) and (50), within the time interval $[t_k, t_{k+1})$, System (5) can be approximated by the following expression:

$$\begin{aligned}\dot{\xi}(t) &= f(x_k) + \frac{\partial f(x_k)}{\partial x}\xi(t) + \frac{1}{2!}\frac{\partial^2 f(x_k)}{\partial x^2}\xi^2(t) \\ &\quad + \left[g(x_k) + \frac{\partial g(x_k)}{\partial x}\xi(t) + \frac{1}{2!}\frac{\partial^2 g(x_k)}{\partial x^2}\xi^2(t)\right]u_k \\ &= \left[f(x_k) + g(x_k)u_k\right] + \left[\frac{\partial f(x_k)}{\partial x} + \frac{\partial g(x_k)}{\partial x}u_k\right]\xi(t) \\ &\quad + \left[\frac{1}{2!}\frac{\partial^2 f(x_k)}{\partial x^2} + \frac{1}{2!}\frac{\partial^2 g(x_k)}{\partial x^2}u_k\right]\xi^2(t) \\ &= \tilde{f}_k + J_k\xi(t) + M_k\xi^2(t)\end{aligned} \quad (51)$$

where

$$\tilde{f}_k = f(x_k) + g(x_k)u_k \quad (52)$$

$$J_k = \frac{\partial f(x_k)}{\partial x} + \frac{\partial g(x_k)}{\partial x}u_k \quad (53)$$

$$M_k = \frac{1}{2}\left(\frac{\partial^2 f(x_k)}{\partial x^2} + \frac{\partial^2 g(x_k)}{\partial x^2}u_k\right) \quad (54)$$

Rewriting Eq. (51), we obtain

$$\dot{x}(t) = \dot{\xi}(t) = \tilde{f}_k + J_k\xi(t) + M_k\xi^2(t), \quad \xi(t_k) = 0 \quad (55)$$

Letting $N > 0$ be an integer number, we denote

$$h = \frac{t_{k+1} - t_k}{N} \quad (56)$$

Using the new step of discretization, h, we replace System (55) by the following equation:

$$\frac{\xi(t_k + (i+1)h) - \xi(t_k + ih)}{h} = \tilde{f}_k + J_k\xi(t_k + ih) + M_k\xi^2(t_k + ih) \quad (57)$$

Therefore,

$$\xi(t_k + (i+1)h) = h\tilde{f}_k + (I + hJ_k)\xi(t_k + ih) + hM_k\xi^2(t_k + ih)$$

where $\xi(t_k + ih)$ are determined recursively by

$$\begin{aligned}&\xi(t_k) = 0 \\ &\xi(t_k + (i+1)h) \\ &\quad = h\tilde{f}_k + (I + hJ_k)\xi(t_k + ih) + hM_k\xi^2(t_k + ih)\end{aligned} \quad (58)$$

Therefore,

$$\begin{aligned}&\xi(t_{k+1}) \\ &= \xi(t_k + Nh) \\ &= h\tilde{f}_k + (I + hJ_k)\xi(t_k + (N-1)h) + hM_k\xi^2(t_k + (N-1)h)\end{aligned} \quad (59)$$

Based on (6), we obtain

$$x_{k+1} = x_k + \xi(t_{k+1}) \quad (60)$$

The above equation gives us a new method to discretize the delay free nonlinear System (5), where $\xi(t_{k+1})$ is determined by (59).

Let us then consider how to apply the new method to the time-delay system (1). Each sampling time interval $[kT,$ should be considered as two subintervals, $[kT, kT+\gamma)$ and $(kT+\gamma, kT+T]$.

In order to apply the discretization method of (60), we should first choose N_1 and N_2, and make them meet the requirement

$$\frac{N_1}{N_2} \approx \frac{\gamma}{T - \gamma} \quad (61)$$

where N_1 and N_2 are both positive natural numbers. Equation (61) indicates that the calculation step lengths of $[kT, kT+\gamma)$ and $(kT+\gamma, kT+T]$ are nearly identical (see also (64) and (70)).

Applying (60) for the subinterval $[kT, kT+\gamma)$, we obtain

$$x_{kT+\gamma} = x_k + \xi(t_k + \gamma) \quad (62)$$

$$\begin{aligned}&\xi(t_k + \gamma) = \xi(t_k + N_1h_1) \\ &= h_1\tilde{f}_{k_1} + (I + h_1J_{k_1})\xi(t_k + (N_1 - 1)h_1) + h_1M_{k_1}\xi^2(t_k + (N_1 - 1)h_1)\end{aligned} \quad (63)$$

where

$$h_1 = \frac{\gamma}{N_1}, \quad (64)$$

$$\tilde{f}_{k_1} = \tilde{f}_k(x(kT), u(k-q-1)), \quad (65)$$

$$J_{k_1} = J_{\xi}(x(kT), u(k-q-1)). \quad (66)$$

$$M_{k_1} = M_{\xi}(x(kT), u(k-q-1)) \quad (67)$$

Applying (60) for the subinterval $(kT+\gamma, kT+T]$, we obtain

$$x_{kT+T} = x_{kT+\gamma} + \xi(t_{kT+\gamma} + (T-\gamma)) \quad (68)$$

$$\begin{aligned}\xi(t_{kT+\gamma} + (T-\gamma)) &= \xi(t_{kT+\gamma} + N_2 h_2) \\ &= h_2\tilde{f}_{k_2} + (I + h_2 J_{k_2})\xi(t_{kT+\gamma} + (N_2-1)h_2) \\ &\quad + h_2 M_{k_2}\xi^2(t_{kT+\gamma} + (N_2-1)h_2) \end{aligned} \quad (69)$$

where

$$h_2 = \frac{T-\gamma}{N_2}, \quad (70)$$

$$\tilde{f}_{k_2} = \tilde{f}_k(x(kT+\gamma), u(k-q)), \quad (71)$$

$$J_{k_2} = J_\xi(x(kT+\gamma), u(k-q)). \quad (72)$$

$$M_{k_2} = M_\xi(x(kT), u(k-q)) \quad (73)$$

By combining (62) and (68), we obtained

$$x_{k+1} = x_k + \xi(t_k+\gamma) + \xi(t_{kT+\gamma} + (T-\gamma)) \quad (74)$$

where $\xi(t_k+\gamma)$ and $\xi(t_{kT+\gamma} + (T-\gamma))$ are calculated according to Eqs. (63) and (68), respectively.

Remark 2: The special case where $\gamma = 0$ and $D = qT$ frequently occurs in practice when modeling and designing digital control systems. In this case, we can use Eq. (60) directly.

3.4 Programming algorithm

The algorithm of the proposed hybrid discretization method for nonlinear time-delay systems is described as follows:

- Given $\frac{dx(t)}{dt} = f(x(t)) + g(x(t))u(t-D)$, the sampling time *T*, time delay *D*, initial state $x(0)$, tolerable error ε, and simulation time *ST*;
- Calculate J_k, $\tilde{f}_k$, M_k, q, γ, $nn = \frac{ST}{T}$, ($u(k) = 0, while$ k<0)
- Perform the loop iterations (loop for $k = 0,1,...,(nn-1)$)
  $u_1(k) = u(k-q-1)$, $u_2(k) = u(k-q)$
  If $\gamma = 0$ do
    a) Calculate $C_k(x = x(k), u = u_2(k))$
    b) $Z_k = T * C_k$
    c) Calculate b^*, then $b = b^* + 3$ (Reference [9])
    d) $N_k = 2^b$; $h_k = \frac{T}{N_k}$
    e) Calculate x_{k+1} (Eqs. (60) and (59))
  Else
    a) Calculate $C_{k_1}(x = x(k), u = u_1(k))$
    b) $Z_k = \gamma * C_k$;
    c) Calculate b^*, then $b = b^* + 3$ (Reference [9])
    d) $N_1 = 2^b$; $h_1 = \frac{\gamma}{N_1}$
    f) $N_2 = \text{int}_+\left(\frac{(T-\gamma)\cdot N_1}{\gamma}\right)$; $h_2 = \frac{T-\gamma}{N_2}$
    e) Calculate $x_{kT+\gamma}$ (Eqs. (62) and (63))
    g) Calculate x_{k+1} (Eqs. (68) and (69))
  End if
- End of loop
- The solution to the discretization of the time-delay system is $x(k)$, $k = 0,1,2,3,...,nn$.

4. Case study

The proposed hybrid time-discretization method for nonlinear control systems with time delay is evaluated through application to two typical systems: a simple first-order process system and a simple analytic second-order system. Various sampling periods and input delays are introduced in the simulation. At the same time, MATLAB ODE solver is used to obtain exact solutions to evaluate the proposed time-discretization method. The values obtained by the proposed method calculated by MAPLE are compared with the results given by the Taylor series method. All of the simulation results show that the proposed method is more accurate than the previous method in [10].

4.1 First-order process system

A simple chemical process system, which is exactly the same system as described in [11, 12], is considered. The system can be described as follows:

$$\frac{dx}{dt} = f(x) + g(x)u = -(1+3a)x + au - ux - ax^2 \quad (75)$$

In this system,

$$f(x) = -(1+3a)x - ax^2$$
$$g(x) = (a-x) \quad (76)$$

We assume that $a = 1.5$ and $x(0) = 0$ in the following simulations.

4.1.1 Unit step input

The following unit step input is applied to the system:

$$u(t-D) = 1.0 \quad (77)$$

when sampling *T=0.25 s* and *D=0.08 s*, the state response and response error are shown in Figs. 1 and 2, respectively.

Fig. 1 shows that the state response of the hybrid discretization method coincides with the results of MATLAB. Fig. 2 shows the error comparison between the hybrid and Taylor methods. The maximum error of hybrid is 6.94e-4. The maximum error of Taylor is 8e-3.

When sampling *T=0.25 s* and *D=0.3 s*, the state response and response error are shown in Figs. 3 and 4, respectively. In this situation, the time delay is bigger than sampling time. The

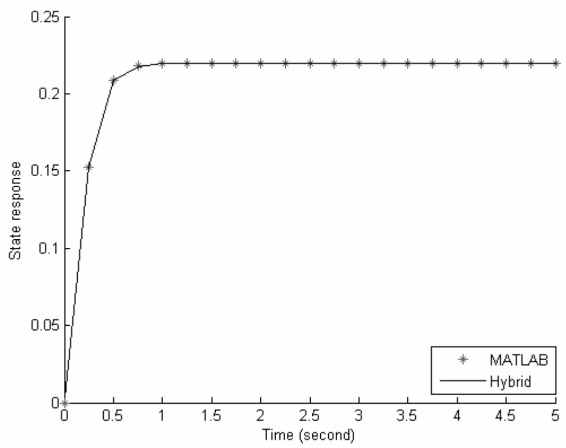

Fig. 1. State response of first-order system ($T=0.25s,\quad D=0.08s$).

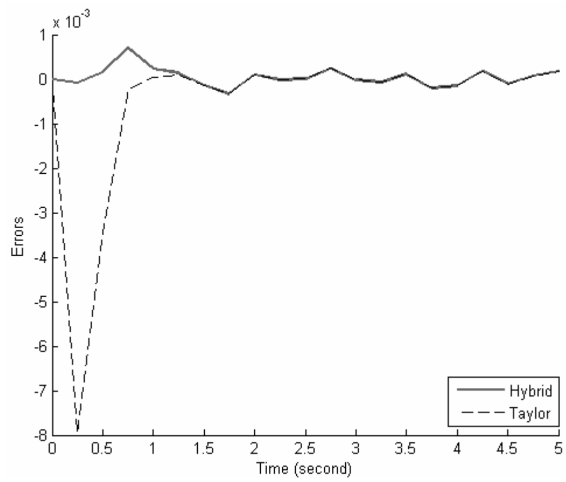

Fig. 2. Error comparison ($T=0.25s, D=0.08s$).

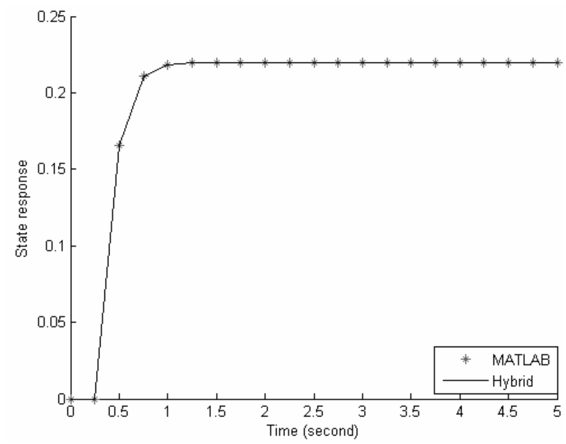

Fig. 3. State response of first-order system ($T=0.25s, D=0.3s$).

maximum errors of the hybrid and Taylor methods are 3.93e-4 and 0.151, respectively.

In the simulations from Figs. 1 to 2, the parameters of the hybrid discretization method are $N_1=128$, $N_2=272$, $h_1=0.00625$, and $h_2=0.00625$. In the simulations from Figs. 3 to 4, the parameters of the hybrid discretization method are $N_1=32$, $N_2=128$, $h_1=0.0015625$, and $h_2=0.0015625$.

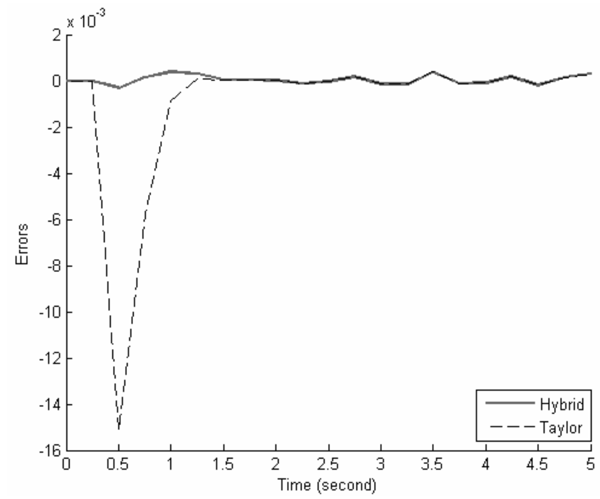

Fig. 4. Error comparison ($T=0.25s, D=0.3s$).

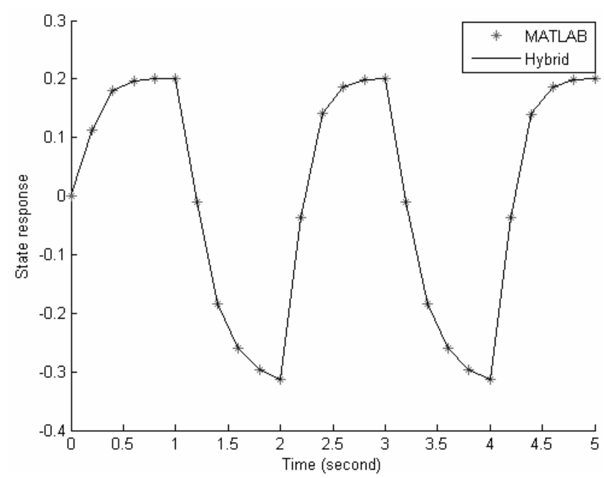

Fig. 5. State response of first-order system ($T=0.2s, D=0.08s$).

4.1.2 Square input

The following unit step input is applied to the system:

$$u(t-D)=0.9square(\pi\cdot(t-D)) \tag{78}$$

when sampling *T=0.2 s* and *D=0.08 s*, the state response and response error comparison are shown in Figs. 5 and 6, respectively.

In the simulations from Figs. 5 to 6, the parameters of the hybrid discretization method are $N_1=128$, $N_2=196$, $h_1=0.00625$, and $h_2=0.00625$. The maximum errors are 0.001 and 0.0475, respectively, for the hybrid and Taylor methods.

When sampling *T=0.1 s* and *D=0.08 s*, the state response and error comparison are shown in Figs. 7 and 8, respectively.

In the simulations from Figs. 7 and 8, the parameters of the

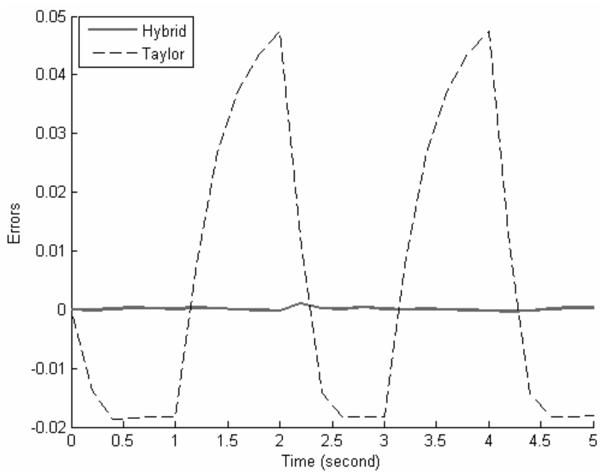

Fig. 6. Error comparison ($T = 0.2s, D = 0.08s$).

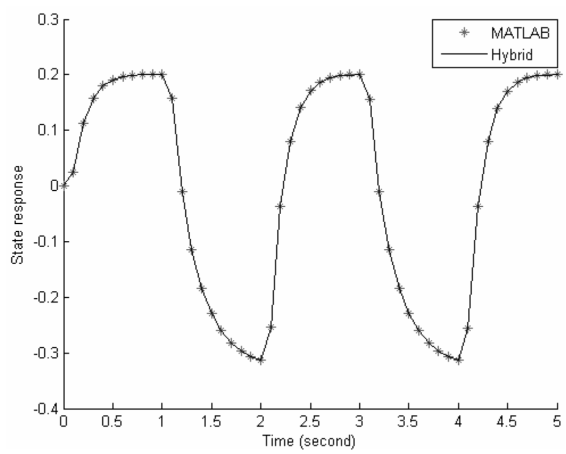

Fig. 7. State response of first-order system ($T = 0.1s, D = 0.08s$).

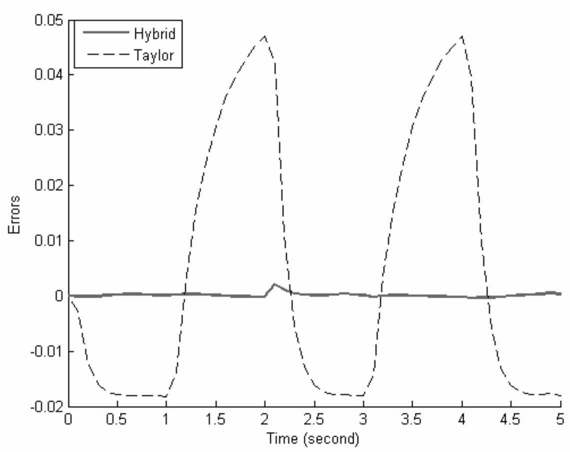

Fig. 8. Error comparison ($T = 0.1s, D = 0.08s$).

proposed method are $N_1 = 128$, $N_2 = 32$, $h_1 = 0.000625$, and $h_2 = 0.000625$. The maximum errors are 0.0020 and 0.0470, respectively, for the hybrid and Taylor methods.

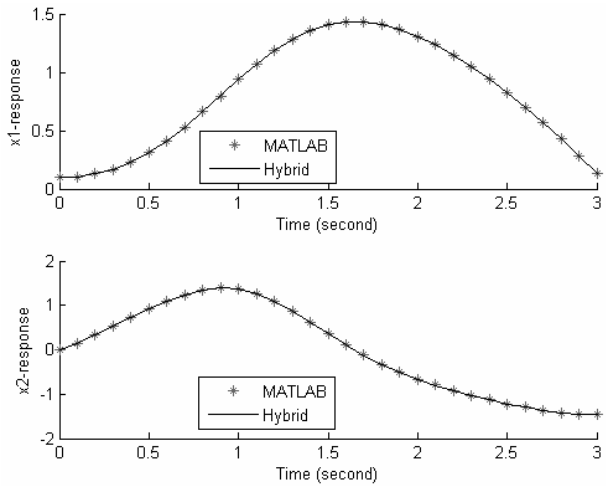

Fig. 9. State response of second-order system ($T = 0.1s, D = 0.02s$).

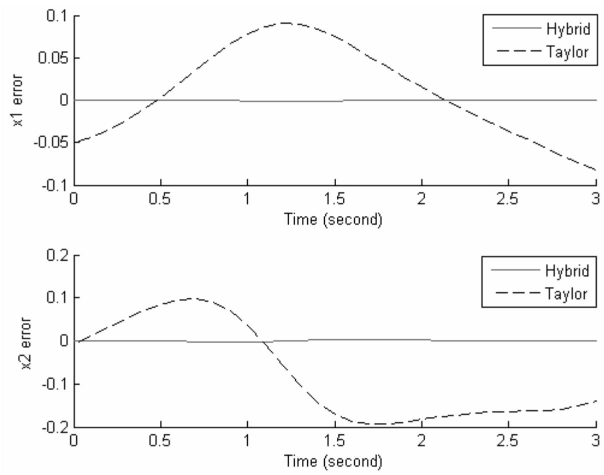

Fig. 10. Error comparison ($T = 0.1s, D = 0.02s$).

4.2 Second-order system

In this section, a single input and single output second-order system is studied. The system is modeled as follows:

$$\ddot{x} = \dot{x}(1 - x^2) - 3x + 2u \tag{79}$$

It is assumed that the initial conditions are $x(0) = 0.1$ and $\dot{x}(0) = 0.0$.

The state variables of this system are defined as follows:

$$X_1 = x, \quad X_2 = \dot{x} \tag{80}$$

Therefore, the state space system model of (51) is as follows:

$$\dot{X}_1 = f_1(X) + g_1(X)u = X_2$$
$$\dot{X}_2 = f_2(X) + g_2(X)u = X_2(1 - X_1^2) - 3X_1 + 2u \tag{81}$$

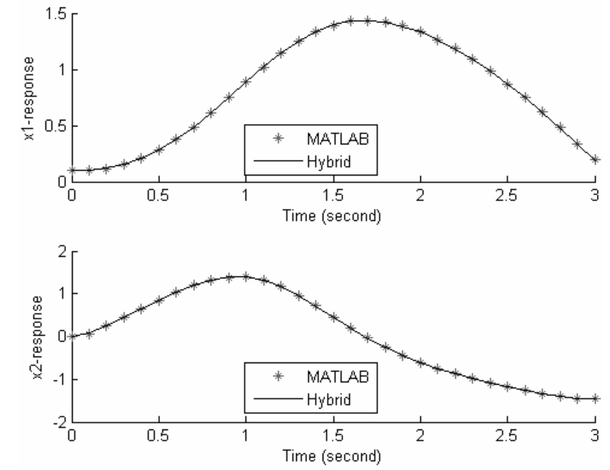

Fig. 11. State response of second-order system ($T = 0.1s, D = 0.05s$).

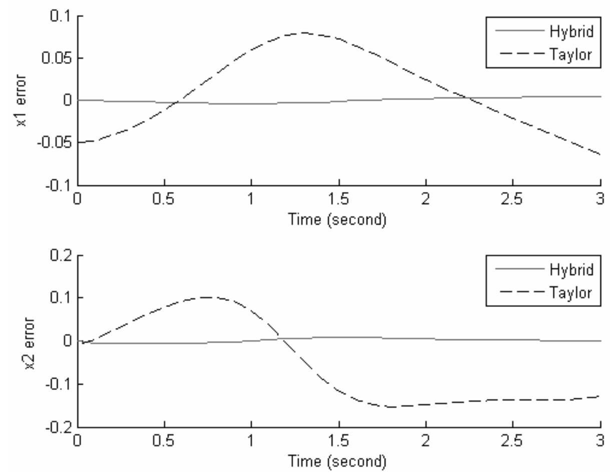

Fig. 12. Error comparison ($T = 0.1s, D = 0.05s$).

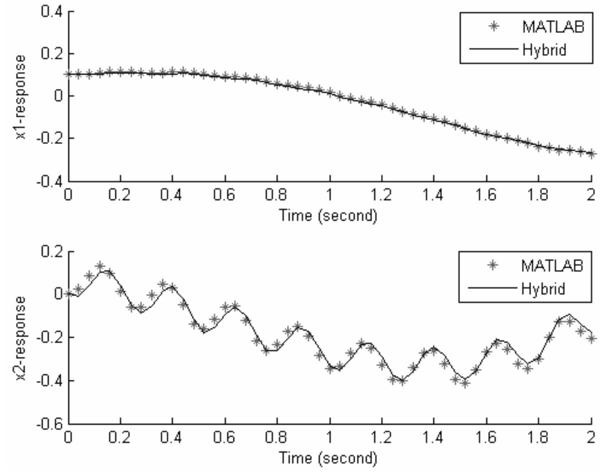

Fig. 13. State response of second-order system ($T = 0.04s, D = 0.002s$).

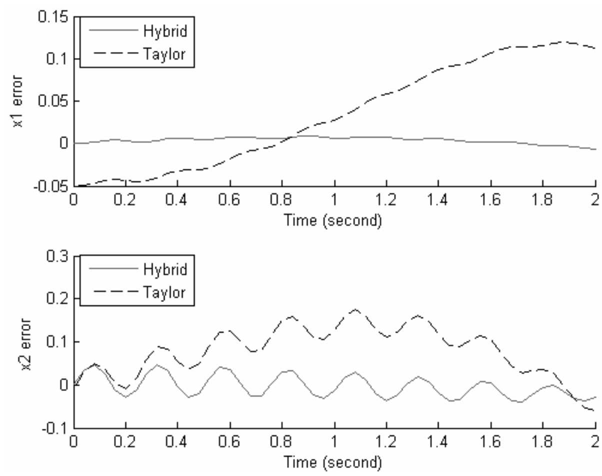

Fig. 14. Error comparison ($T = 0.04s, D = 0.002s$).

4.2.1 Unit step input

The following unit step input is applied to the system,

$$u(t - D) = 1.0 \tag{82}$$

When sampling *T=0.1 s* and *D=0.02 s*, the state response and error comparison are shown in Figs. 9 and 10, respectively.

In the simulations from Figs. 9 to 10, the parameters of the proposed method are $N_1 = 32$, $N_2 = 128$, $h_1 = 0.000625$, and $h_2 = 0.000625$. The maximum errors are 0.0074 and 0.0514 for X_1 and X_2 , respectively, for the proposed method. The maximum errors of the Taylor method are 0.0904 and 0.1936 for X_1 and X_2 , respectively.

When sampling *T=0.1 s* and *D=0.05 s*, the state response and error comparison are shown in Figs. 11 and 12, respectively.

In the simulations from Figs. 11 to 12, the parameters of the proposed method are $N_1 = 256$, $N_2 = 256$, $h_1 = 0.000195$, and $h_2 = 0.000195$. The maximum errors of the proposed method are 0.0040 and 0.0084 for X_1 and X_2 , respectively. The maximum errors of the Taylor method are 0.079 and 0.1526 for X_1 and X_2 , respectively.

4.2.2 Sine input

The following sine input is applied to the system:

$$u(t - D) = \sin(2\pi \cdot 4 \cdot (t - D)) \tag{83}$$

When sampling *T=0.04 s* and *D=0.002 s*, the state response and response error are shown in Figs.13 and 14, respectively.

In the simulations from Figs. 13 to 14, the parameters of the proposed method are $N_1 = 8$, $N_2 = 152$, $h_1 = 0.00025$, and $h_2 = 0.00025$. The maximum errors of the proposed method are 0.0087 and 0.0474 for X_1 and X_2 , respectively. The maximum errors of the Taylor method are 0.1193 and 0.1742, respectively.

When sampling *T=0.03 s* and *D=0.002 s*, the state response and response error are shown in Figs. 15 and 16, respectively.

In the simulations from Figs. 15 to 16, the parameters of the proposed method are $N_1 = 8$, $N_2 = 92$, $h_1 = 0.00025$, and $h_2 = 0.00025$. The maximum errors are 0.0039 and 0.0282 for

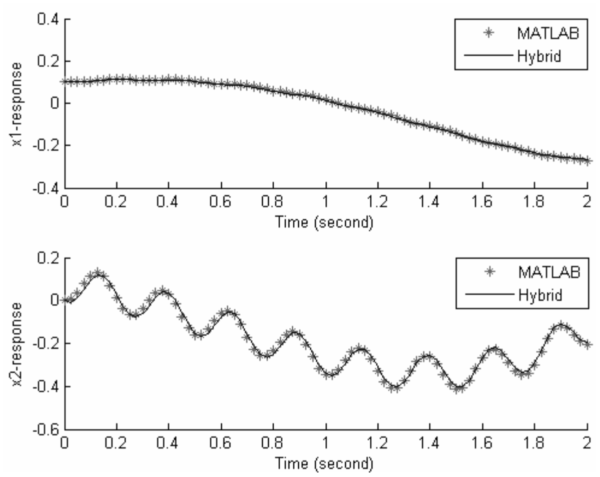

Fig. 15. State response of second-order system ($T = 0.03s, D = 0.002s$).

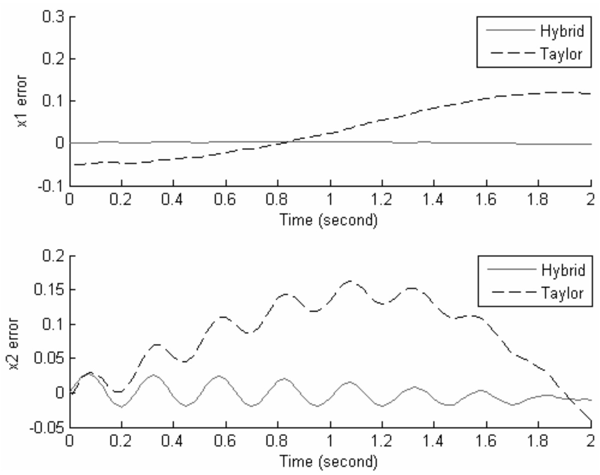

Fig. 16. Error comparison ($T = 0.03s, D = 0.002s$).

X_1 and X_2, respectively. The maximum errors of the Taylor method are 0.1206 and 0.1616, respectively.

5. Conclusion

A hybrid discretization method based on a combination of the Taylor series with the Matrix exponential is proposed for time-delay nonlinear systems. The mathematical structure of the new discretization scheme was explored and some case studies were presented. The proposed time-discretization method provides a finite-dimensional representation for nonlinear control systems with time delay.

The performance of the proposed time-discretization procedure was evaluated using two case studies with increasing complexity: a first-order process control system and a second-order system. Various sampling rates and time-delay values were considered in the sample studies. The results of the simulation were compared with those given by MATLAB, in order to verify the accuracy of the proposed method. These examples demonstrate the use of the proposed method to solve a real system. In these cases, even when the sampling time is large with input time delay, the hybrid method can meet the accuracy requirement of the systems.

The simulation results show that the proposed method has the following virtues:

It is suitable for any nonlinear input time-delay system, especially for large sampling period systems with large time delay.

It is easy to program for use in a nonlinear input time-delay system using the hybrid discretization method.

It can easily be inserted into other larger control programs.

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